TABLE II. Numerical evaluation of criteria on the validity of the suggested $\{\omega,\phi,\rho,B\}$ resonance model for nucleon electromagnetic structure. The left-hand sides of Eqs. (21), (22), and (24) are numerically tabulated for this model.

Equation number	Numerical result	Theoretical constraint
(21) Vector case	(-0.016 ± 0.022) F^2	$=0$
(21) Scalar case	(0.199 ± 0.687) F^2	$=0$
(22) $\lim_{q^2\to\infty}(q^2G_E^p)$	(-6.36 ± 1.46) F^{-2}	$\geqslant 0$
(24) $\lim_{q^2\to\infty}(q^2G_M^p)$	(4.68 ± 1.46) F^{-2}	$\geqslant 0$

Within the experimental errors, the model is not inconsistent with the threshold condition, Eq. (21), and the positivity condition on $\lim_{q^2\to\infty}(q^2G_M^p)$, Eq. (24). However, the positivity restriction on $\lim_{q^2\to\infty}(q^2G_E^p)$, Eq. (22), is badly violated. It should be noted that this latter condition is completely independent of the threshold condition. Although the effective masses of the vector particles may be shifted slightly due to their broad widths,[15] this effect (or the experimental uncertainties in the masses) does not significantly modify the above results. Consequently, we conclude that the $\{\omega,\phi,\rho,B\}$ resonance model cannot accommodate all of the most evident general features of the experimental data on nucleon electromagnetic form factors.

ACKNOWLEDGMENTS

We are grateful to Professor A. Bincer for a valuable suggestion on the proof of the threshold condition. We thank Professor A. Bincer, Professor C. Goebel, and Professor R. F. Sawyer for carefully reading the manuscript.

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PHYSICAL REVIEW VOLUME 136, NUMBER 1B 12 OCTOBER 1964

Particle Mixing and Renormalization*

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(Received 5 May 1964)

The field theory of neutral vector particles interacting with conserved currents is investigated as an example of particle mixing. It is shown that a generalization of conventional renormalization is necessary when mixing occurs, and that the observable masses and coupling constants are sufficient to determine transition amplitudes, without recourse to mixing parameters. The universality of electric charge renormalization is not changed when photon–vector-meson mixing is possible.

INTRODUCTION

THEORETICAL and experimental physicists are currently investigating mixing between particles of the same spin, parity, charge, and baryon number.[1–4] It is the aim of this paper to show how a sound theoretical basis might be given, from which the consequences of mixing could be predicted. In the belief that it is the most interesting and physically relevant case, we confine the discussion to the mixing of neutral vector particles, such as the photon and φ, ρ, ω mesons, which is caused by their interactions with conserved currents (which we shall assume renormalizable).

In the main part of the paper we show how mixing may be correctly taken into account by an extension of conventional renormalization, and we then consider photon-vector-meson mixing as a particular case.

VECTOR-PARTICLE FIELD THEORY

We can use covariant notation[5] to write the Lagrangian density[6] for neutral vector fields A_ν^i, $i=1,\cdots n$, interacting with conserved currents J_μ^α, $\alpha=1,\cdots,N$, in a particular but arbitrary gauge specified by constants λ_i. This method allows us to consider massive and massless particles together; for the latter, we shall put in a mass M, and take the limit $M\to 0$ at the appropriate place.

$$L=L_0[A^i,m_i,\lambda_i]-\sum_\alpha g_{i\alpha}A_\mu{}^iJ_\mu{}^\alpha + \text{terms not involving the } A\text{'s}, \quad (2.1)$$

* The research reported in this document has been sponsored in part by the Air Force Office of Scientific Research OAR through the European Office Aerospace Research United States Air Force.

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[6] We use the notation $a_\mu b_\mu \equiv a_0b_0-\mathbf{a}\cdot\mathbf{b}$ and $\partial_\mu\equiv(\partial/\partial x_0, -\partial/\partial\mathbf{x})$. Repeated indices i, j, k are summed over, but the repeated index α is only summed over when Σ_α precedes the expression involved.

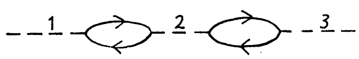

FIG. 1. Feynman diagram with off-diagonal poles.

where

$$L_0[A^i,m_i,\lambda_i]\equiv -\tfrac{1}{2}[\partial_\mu A_\nu{}^i\partial_\mu A_\nu{}^i-\partial_\mu A_\nu{}^i\partial_\nu A_\mu{}^i]+\tfrac{1}{2}m_i^2[A_\mu{}^iA_\mu{}^i-\partial_\mu A_\mu{}^i\partial_\nu A_\nu{}^i/\lambda_i^2].$$

The constants m_i and $g_{i\alpha}$ will be determined from the observed masses and coupling constants in the next section, although the constants $g_{i\alpha}$ might be restricted by some underlying theory.

We can quantize the theory in the usual way, and assuming a perturbation solution is valid, calculate matrix elements by considering Feynman diagrams. The free propagator for the kth field can be written $g_{\mu\nu}/(p^2-m_k^2)$, since current conservation ensures that there is no contribution from terms involving $p_\mu p_\nu$.

In order to obtain finite contributions from individual diagrams we must define renormalized fields and introduce the physical masses and coupling constants. This procedure will also simplify the structure of the total propagator, which before renormalization is a matrix Δ^{ij} with poles in each element (ij). For example the diagram in Fig. 1 will contribute poles at $p^2=m_1^2$, m_2^2, and m_3^2 to the fourth-order propagator.

RENORMALIZATION

The renormalized fields $\tilde{A}_\nu{}^i$ are defined by a linear transformation[7]:

$$A_\nu{}^i=R_{ik}\tilde{A}_\nu{}^k. \tag{3.1}$$

The Lagrangian can then be rewritten in terms of constants to be defined below:

$$L=L_0[\tilde{A}^i,M_i,\lambda_i]+L_{\text{int}}+\text{terms not involving the } A\text{'s}, \tag{3.2}$$

where

$$L_{\text{int}}=-\textstyle\sum_\alpha \tilde{g}_{i\alpha}(1-L_{i\alpha})\tilde{A}_\mu{}^i\tilde{J}_\mu{}^\alpha -\tfrac{1}{2}\partial_\mu\tilde{A}_\nu{}^iC_{ij}\partial_\mu\tilde{A}_\nu{}^j-\tfrac{1}{2}\tilde{A}_\mu{}^iD_{ij}\tilde{A}_\mu{}^j +\text{terms of the form } \partial_\mu\cdots\partial_\nu\cdots.$$

The renormalized currents $\tilde{J}_\mu{}^\alpha=Z_{2\alpha}{}^{-1}J_\mu{}^\alpha$ are defined in the usual way, and we define

$$R_{ki}R_{kj}=\delta_{ij}+C_{ij}, \tag{3.3}$$

$$R_{ki}m_k^2R_{kj}=\delta_{ij}M_{(i)}{}^2-D_{ij}, \tag{3.4}$$

$$g_{i\alpha}=(Z_1)_{k\alpha}Z_{2\alpha}{}^{-1}R^{-1}{}_{ki}\tilde{g}_{k\alpha}, \tag{3.5}$$

$$(Z_1)_{k\alpha}=1-L_{k\alpha}. \tag{3.6}$$

We shall now see that if C, D, L are defined correctly, $\tilde{g}_{i\alpha}$ and M_i are the physical coupling constants and masses, and we can then regard Eqs. (3.3)–(3.6) as determining the bare constants $g_{i\alpha}$, m_i in terms of the physical quantities $\tilde{g}_{i\alpha}$, M_i.

[7] We could have written R_{ik} as $(Z_3{}^{1/2})_{ik}$ to emphasize that this is only a generalization of Z_3 renormalization.

We can again obtain a perturbation solution, taking L_{int} as the interaction Lagrangian. Neglecting $p_\mu p_\nu$ terms, we have free propagators $g_{\mu\nu}/(p^2-M_k^2)$, $(i\alpha\alpha)$ vertices $\tilde{g}_{i\alpha}(1-L_{(i)\alpha})\gamma_\nu$, and (ij) vertices $-g_{\mu\nu}(D_{ij}+p^2C_{ij})$. We can now define C, D, L so that the resulting proper vertex correction $\tilde{g}_{i\alpha}\Lambda_{F\nu}{}^{(i)\alpha}(p,q)$ and proper self-energy/mixing insertion $\Pi_{F\mu\nu}{}^{ij}(p)$ are both finite, with the following boundary conditions:

$$\Lambda_{F\nu}{}^{i\alpha}(p,q)=0 \quad\text{when}\quad p^2=q^2=M_\alpha^2,\ (p-q)^2=M_i^2 \tag{3.7}$$

(between appropriate wave functions),

$$\Pi_{F\mu\nu}{}^{ij}(p)=0 \quad\text{when}\quad p^2=M_i^2 \quad\text{or}\quad p^2=M_j^2 \tag{3.8}$$

(double zero if $i=j$).

It can then be verified that L and $D+p^2C$ are the true divergent parts of the vertex and self-energy corrections, when these are calculated by making the appropriate subtractions at each intermediate stage instead of using counter terms (C,D,L). These vertex and self-energy corrections, which are merely linear combinations of those which would be obtained from the unrenormalized theory, can thus be written

$$\Lambda_\nu{}^{i\alpha}(p,q)=L_{i\alpha}\gamma_\nu+\Lambda_{F\nu}{}^{i\alpha}(p,q), \tag{3.9}$$

$$\Pi_{\mu\nu}{}^{ij}(p)=(g_{\mu\nu}-p_\mu p_\nu/p^2)[D_{ij}+p^2C_{ij}+(p^2-M_i^2)(p^2-M_j^2)\times T_{ij}(p^2)], \tag{3.10}$$

when we include the $p_\mu p_\nu$ terms required by gauge invariance.

Equation (3.8) implies that the total effective propagator can be written in the form

$$\Delta_{\mu\nu}{}^{ij}(p)=g_{\mu\nu}[\delta_{ij}/(p^2-M_{(i)}{}^2)+\text{terms finite at } p^2=M_k^2],\ k=1,\cdots,n. \tag{3.11}$$

It is to this end that we made the linear transformation, Eq. (3.1). This form for the propagator shows that M_i can be taken as the physical masses of the asymptotic states, and as the masses of intermediate (virtual) states. Also, Eq. (3.7) shows that $\tilde{g}_{i\alpha}$ are the observed coupling constants; the vertex correction vanishes when the momenta are on the appropriate mass shells.

Thus, we have a clear procedure for calculating matrix elements. Draw Feynman diagrams with propagators $g_{\mu\nu}/(p^2-M_k^2)$, and vertices $\tilde{g}_{i\alpha}\gamma_\nu$, each line corresponding to one particle only. Self-energy parts are subtracted twice, once at each relevant mass ($p^2=M_i^2$ and $p^2=M_j^2$), to obtain finite expressions. It is easily seen that corrections to external lines vanish identically. As an example we show the lowest order diagrams for fermion-fermion scattering, with the diagrams giving the lowest order correction to the vector propagator. Summation over i, j, β is implied. The contribution of these diagrams is $\tilde{g}_{i\alpha}\tilde{g}_{i\alpha'}/(p^2-M_i^2)+\tilde{g}_{i\alpha}\tilde{g}_{j\alpha'}T_{ij}{}^{(2)}(p^2)$. Equivalently, if we draw the reduced (Dyson) diagrams for any process, the total (matrix) propagator $\Delta_{\mu\nu}{}^{ij}(p)$ has poles in its diagonal terms only. Figure 3 shows the lowest order Dyson diagrams for the same process, giving a contribution $\tilde{g}_{i\alpha}\tilde{g}_{j\alpha'}\Gamma^{i\alpha}\Delta_{ij}\Gamma^{j\alpha'}$. The only pole terms which occur in the complete amplitude are $\tilde{g}_{i\alpha}\tilde{g}_{i\alpha'}/(p^2-M_i^2)$. Therefore, if we only require the pole terms, we need only consider the first diagram of Fig. 2, and do not need diagrams of the type shown in Fig. 3 with off-diagonal propagators. Thus we have shown that off-diagonal pole terms in the unrenormalized propagator can be incorporated in renormalized coupling constants and masses, and that these coupling constants and masses are the finite observed quantities.

Finally, to obtain the usual vector-meson and photon states, we must take the limit $\lambda_k \to \infty$ for the former and $M_1 \to 0$, $\lambda_1/M_1 \to 1$ for the photon ($i=1$). Matrix elements are obviously continuous in these limits, apart from the well-known infrared divergences which occur when transition amplitudes are calculated.

From the first diagram of Fig. 2 we can obtain the proton form factor in the presence of a vector meson. With obvious notation, the matrix element is $\tilde{e}^2/p^2+\tilde{g}_p\tilde{g}_e/(p^2-M_v^2)$, and hence the form factor is $\tilde{e}+p^2\tilde{g}_p\tilde{g}_e/\tilde{e}(p^2-M_v^2)$, which is the Clementel-Villi[1,8] formula in terms of observable quantities. $\tilde{g}_e$ is to be interpreted as the coupling constant between the electron and the true physical meson, whether or not the bare coupling constant g_e of the bare meson is zero.

THE MIXING PARAMETERS

We have found that the bare fields are linear combinations of the fields defining the true one-particle states, and thus the coefficients R_{ik} are the mixing parameters of the theory. However, they do not appear in matrix elements in the method outlined above, and seem to be infinite if calculated by perturbation theory.

If one assumes particular values for the bare coupling constants, such as are given by unitary symmetry, it would be possible to find the mixing parameters by measuring the physical coupling constants, since $\tilde{g}_{i\alpha}=R_{ki}g_{k\alpha}$ (ignoring Z_1 and Z_2 renormalizations). For example, if the bare fields represented pure i-spin states, one could obtain an estimate of the deviation of the physical states from pure i-spin states.

Only in the most drastic approximation is it possible to take (R) unitary and to write $|i$, physical$\rangle = R_{ki}|k$, bare$\rangle$. In fact the physical one-particle states contain all possible many-particle bare states; for this reason it is simpler to work with fields instead of states, remembering that it is the renormalized fields which define the physical states:

$$\langle 0|\tilde{A}^i(0)|k \text{ physical}\rangle = \delta_{ik}. \qquad (4.1)$$

However, at the moment it appears necessary to make such an approximation in order to calculate the mixing parameters.[9]

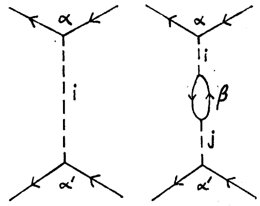

FIG. 2. Feynman diagrams for fermion-fermion scattering.

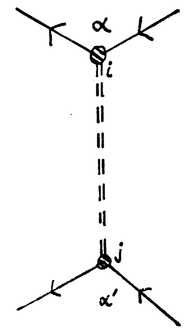

FIG. 3. Dyson diagram for fermion-fermion scattering.

THE PHOTON AND CHARGE RENORMALIZATION

Let us consider the particular case in which A^1 is the photon field, and take the limit $M_1=0$.

The proper self-energy, Eq. (3.10), must be regular at $p^2=0$, and hence

$$D_{ij}=0 \quad \text{if} \quad i=1 \quad \text{or} \quad j=1. \qquad (5.1)$$

It then follows from Eqs. (3.3) and (3.4) that

$$(R^{\pm 1})_{k1}=0, \quad k\neq 1, \qquad (5.2)$$

and

$$m_1=0. \qquad (5.3)$$

These results were also obtained in Ref. 1 using the one-particle approximation. Both the renormalized and the bare photon fields are massless, and the renormalized massive fields have no contribution from the bare photon field. We see from this that a gauge transformation on the bare photon field will not change the massive fields.

It would seem possible that the existence of vector mesons, coupled to the proton but not to the electron, might lead to different renormalized electron and proton charges (if we assume that the bare charges are equal).[10] However, we shall show that as a consequence of Eq. (5.2) and Ward's identity, we obtain the same renormalized charges.

Using unrenormalized Heisenberg fields, the total fermion propagator is given by

$$_0\langle T\psi_\alpha(x)\bar{\psi}_\alpha(y)\rangle_0 = iS_\alpha(x-y). \qquad (5.4)$$

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We can define the vertex function $\Gamma_\nu^{i\alpha}$ by

$$\sum_\beta {}_0\langle T\psi_\alpha(x)g_{i\beta}J_\mu{}^\beta(z)\bar{\psi}_\alpha(y)\rangle_0 = -\int\int d^4u d^4u' S_\alpha(x-u) g_{i\alpha}\Gamma_\mu{}^{(i\alpha)}(u-z;\, z-u')S_\alpha(u'-y). \quad (5.5)$$

Taking the divergence and then the Fourier transform of this equation, we obtain the Ward-Takahashi identity[11,12] for the field A^i interacting with the current J^α:

$$S_\alpha{}^{-1}(p)-S_\alpha{}^{-1}(q)=(p-q)_\nu\Gamma_\nu{}^{i\alpha}(p,q), \quad (5.6)$$

and thus,

$$\partial S_\alpha{}^{-1}(p)/\partial p_\nu=\Gamma_\nu{}^{i\alpha}(p,p). \quad (5.7)$$

The renormalized functions $\tilde{\Gamma}_\nu{}^{i\alpha}$, $\tilde{S}_\alpha$ are given by

$$\tilde{S}_\alpha=Z_{2\alpha}{}^{-1}S_\alpha, \quad (5.8)$$

$$g_{i\alpha}\Gamma_\nu{}^{(i\alpha)}=Z_{2\alpha}{}^{-1}R^{-1}{}_{ki}\tilde{g}_{k\alpha}\tilde{\Gamma}_\nu{}^{k\alpha}. \quad (5.9)$$

Therefore,

$$Z_1{}^{-1}{}_{i\alpha}\tilde{\Gamma}_\nu{}^{(i\alpha)}(p,p)=Z_{2\alpha}{}^{-1}\partial\tilde{S}_\alpha{}^{-1}/\partial p_\nu, \quad (5.10)$$

but

$$\tilde{\Gamma}_\nu{}^{i\alpha}(p^2=q^2=M_\alpha{}^2,\,(p-q)^2=M_i{}^2) = (1+L_{i\alpha})\gamma_\nu=(2-Z_{1i\alpha})\gamma_\nu \quad (5.11)$$

and

$$\partial\tilde{S}_\alpha{}^{-1}/\partial p_\nu(p^2=M_\alpha{}^2)=(1-B_{i\alpha})\gamma_\nu=(2-Z_{2\alpha})\gamma_\nu; \quad (5.12)$$

therefore,

$$\text{if } M_i=0,\quad Z_{1i\alpha}=Z_{2\alpha}. \quad (5.13)$$

Then, from Eqs. (3.5) and (5.2), we deduce that

$$\tilde{g}_{1\alpha}=R_{11}g_{1\alpha},\quad \alpha=1,\cdots,N. \quad (5.14)$$

This shows that the electric charges of all particles are changed by the same factor when renormalized, as we asserted.

CONCLUSION

We have shown that when mixing occurs, the renormalized fields must be taken as linear combinations of the bare fields. In order to calculate matrix elements, observed masses and coupling constants are used, but mixing parameters are not needed. We use a propagator whose pole terms are diagonal, and subtract self-energy parts at each relevant mass in order to calculate it. To lowest order this method gives the same results as the prescription of Feldman and Matthews,[1] and also calculates higher order corrections correctly.

Finally we have shown that photon–vector-meson mixing still allows a zero bare mass for the photon, and that in such circumstances the electric charges of different particles are again renormalized by the same factor.

ACKNOWLEDGMENTS

The author would like to thank Professor P. T. Matthews for his suggestion of the problem and constant attention, and J. Poston for his stimulating comments.

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PHYSICAL REVIEW VOLUME 136, NUMBER 1B 12 OCTOBER 1964

General Relativistic Instability*

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(Received 25 May 1964)

The point of instability of a general relativistic fluid sphere is determined using the criterion that the point of instability occurs when the binding energy is at a maximum. The result is equivalent to the result obtained by the small-perturbation method when the radius is sufficiently large.

A RECENT paper[1,2] has shown, using the method of small perturbations, that gaseous masses may exhibit a radial instability in the framework of general relativity. When discussing instability in Newtonian physics, one sometimes uses an energy method to determine the point of instability, and it can be shown that in certain cases the methods are equivalent. For the case of a general relativistic fluid sphere with constant energy density and heat capacity, one can show that the two methods give exactly equivalent results at the limit where the radius is much greater than the Schwarzschild radius. The instability is assumed to occur when the binding energy of the system is at a maximum.

The expression for the binding energy of a static,

* Sponsored by the Mathematics Research Center, U. S. Army, Madison, Wisconsin, under Contract No. DA-11-022-ORD-2059.
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